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COMMENT

Fractional dimension of sets in discrete spaces

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Abstract. We give a new definition $\dim_H(A)$ for the dimension of an arbitrary subset of the lattice $\mathbb{Z}^d$. We establish elementary properties, and calculate the dimension for some examples. Finally, we announce a result which states that, if $\dim_H(A) < d-2$, then A is transient for the simple random walk on $\mathbb{Z}^d$, and that if $\dim_H(A) > d-2$ then A is recurrent.

1. Introduction

Although Hausdorff dimension is the most commonly used definition of dimension for subsets of $\mathbb{R}^d$, there are contexts where some other definition is more appropriate [1, 2]. Any reasonable definition will give the same answer for strictly self-similar sets, but even for affinely self-similar sets different definitions may give different values [3].

The definitions of dimension for subsets of $\mathbb{R}^d$ all relate to the microscopic (i.e. local) properties of the set. However, many models in statistical physics involve working on a lattice (such as $\mathbb{Z}^d$), and any definition of dimension here must be related to the global properties of the set. One such definition based on the 'mass' of the set is in common use. Let $V(0, n)$ denote the cube with centre 0 and side n, and for a set A in $\mathbb{Z}^d$ set

$$\begin{aligned}\dim_{UM}(A) &= \limsup_{n\to\infty} \frac{\ln|A \cap V(0,n)|}{\ln n} \\ \dim_{LM}(A) &= \liminf_{n\to\infty} \frac{\ln|A \cap V(0,n)|}{\ln n}.\end{aligned} \tag{1}$$

If these two numbers agree we call their common value the mass dimension of A and write it as $\dim_M(A)$; otherwise we refer to $\dim_{UM}(A)$, $\dim_{LM}(A)$ as respectively the upper and lower mass dimensions of A. (It is easy to check that these numbers do not depend on the choice of 0 as the 'base point', and that the limits in (1) have the same value if $n \to \infty$ through a subsequence $n_k = 2^k$.)

The mass dimension does seem to be useful in a great many contexts. However, as in the case of $\mathbb{R}^d$, one might expect there to be occasions when other definitions are more suitable. In this comment we give a new definition of dimension: we believe it to be the 'correct' lattice analogue of Hausdorff dimension. A different proposed definition has been given in [4], but, as we explain below, this has some undesirable properties.

2. The definition and elementary properties

We begin by setting up some notation. If $A \subseteq \mathbb{Z}^d$ then the sets λA, $A+x$ are defined by $\lambda A = \{\lambda x : x \in A\}$, $A+x = \{y+x : y \in \mathbb{Z}^d\}$. For a point $x = \{x_1, \ldots, x_d) \in \mathbb{Z}^d$, and $n \geq 1$ we set

$$C(x, n) = \{y \in \mathbb{Z}^d : x_i \leq y_i < x_i + n\}$$

$$V(x, n) = \{y \in \mathbb{Z}^d : x_i - \tfrac{1}{2}n \leq y_i < x_i + \tfrac{1}{2}n\}.$$

We call $C(x, n)$ the cube with base x and side n, and $V(x, n)$ the cube with centre x and side n. Note that $C(x, 1) = V(x, 1) = \{x\}$, and that $|C(x, n)| = |V(x, n)| = n^d$. ($|A|$ denotes the number of points in A.) All the cubes we consider have sides parallel to the axes. If B is a cube we denote by $s(B)$ the length of the side of B.

We will also need a subcollection of cubes, the *dyadic* cubes. A cube B is a dyadic cube if B is of the form $C(x, 2^n)$, where $x \in 2^n\mathbb{Z}^d$. If B_1, B_2 are dyadic cubes then either one is contained in the other, or B_1 and B_2 are disjoint. So, any cover of a set by dyadic cubes has a subcover of disjoint dyadic cubes. Let $S_1 = V(0, 2)$, and for $m \geq 2$ set

$$S_m = V(0, 2^m) \setminus V(0, 2^{m-1}). \tag{2}$$

Thus (S_m) is a sequence of disjoint cubical shells centred on the point $(-\tfrac{1}{2}, -\tfrac{1}{2}, \ldots, -\tfrac{1}{2})$.

Let $h: [0, \infty) \to [0, \infty)$ be a strictly increasing function with $h(0) = 0$. For $A \subseteq \mathbb{Z}^d$, $n \geq 1$, set

$$\nu_h(A, 2^n) = \min_{A \cap S_n \subseteq \bigcup_{i=1}^m B_i} \sum_{i=1}^m h\left(\frac{s(B_i)}{2^n}\right) \tag{3}$$

where the minimum is taken over all covers of $A \cap S_n$ by any set of cubes $B_1, \ldots, B_m$ of the form $C(x, k)$. Now set

$$m_h(A) = \sum_{n=1}^{\infty} \nu_h(A, 2^n). \tag{4}$$

Let $\tilde{\nu}_h(A, 2^n)$ and $\tilde{m}_h(A)$ be defined in the same way, except that the minimum is taken over covers by dyadic cubes. If $h(x) = x^\alpha$ we write ν_α, m_α for ν_h, m_h. We now define

$$\dim_H(A) = \sup\{\alpha : m_\alpha(A) = \infty\}. \tag{5}$$

We will call $\dim_H(A)$ the *discrete Hausdorff dimension* of A.

We now list some elementary properties of the definitions.

(i) By taking $B_1 = V(0, 2^n)$ we have $\nu_h(A, 2^n) \leq h(1)$ for any set A and $n \geq 1$. It follows that ν_h and m_h only depend on $h(x)$ for $0 \leq x \leq 1$.

(ii) If $h_1 \leq h_2$ then it is clear that $\nu_{h_1}(A, 2^n) \leq \nu_{h_2}(A, 2^n)$, and so that $m_{h_1}(A) \leq m_{h_2}(A)$. Let $\alpha \leq \beta$: then since $x^\alpha \geq x^\beta$ for $0 \leq x \leq 1$ we have $m_\alpha(A) \geq m_\beta(A)$. Thus $m_\alpha(A) = \infty$ for $\alpha < \dim_H(A)$, and $m_\alpha(A) < \infty$ for $\alpha > \dim_H(A)$.

(iii) It is clear that $\tilde{\nu}_h(A, 2^n) \geq \nu_h(A, 2^n))$. If $B_1, \ldots, B_m$ is an optimal cover of $A \cap S_n$ by cubes, then there exist dyadic cubes Q_{ij}, $1 \leq i \leq m$, $1 \leq j \leq 2^d$, such that each $b_i \subseteq \bigcup_{j=1}^{2^d} Q_{ij}$, and $s(Q_{ij}) \leq s(B_i)$. So

$$\tilde{\nu}_h(A, 2^n) \leq 2^d \nu_h(A, 2^n) \tag{6}$$

and thus $\tilde{m}_h(A) < \infty$ if and only if $m_h(A) < \infty$.

(iv) If $A_1 \subseteq A_2$ then $\nu_h(A_1, 2^n) \leq \nu_h(A_2, 2^n)$. So $m_h(A_1) \leq m_h(A_2)$ and $\dim_H(A_1) \leq \dim_H(A_2)$.

(v) If A is finite then $A \cap S_n$ is empty for all large n, and so $\nu_h(A, 2^n) = 0$ for all large n. Thus $m_h(A) < \infty$ for any h, and in particular $\dim_H(A) = 0$.

(vi) For any set $A \subseteq \mathbb{Z}^d$ we have $\dim_H(A) \leq \dim_{UM}(A)$: the discrete Hausdorff dimension is less than the upper mass dimension. To see this, write $a_n = |A \cap V(0, 2^n)|$, $\gamma = \dim_{UM}(A)$. It is easily checked that $\gamma = \limsup_{n\to\infty} \ln a_n / \ln(2^n)$, so that if $\alpha > \beta > \gamma$ then $a_n \leq 2^{n\beta}$ for all large n. Covering $A \cap V(0, 2^n)$ by a_n cubes of side 1 we have

$$\nu_\alpha(A, 2^n) \leq a_n 2^{-n\alpha} \leq 2^{-n(\alpha-\beta)}.$$

So $m_\alpha(A) < \infty$, and $\dim_H(A) \leq \gamma$.

(vii) For any $A \subseteq \mathbb{Z}^d$, $0 \leq \dim_H(A) \leq d$. This is immediate from (iv), (v) and (vi).

(viii) If $A \subseteq \mathbb{Z}^d$ and $B = x + A$ (so that B is A translated by x) then $m_h(A) < \infty$ if and only if $m_h(B) < \infty$. In particular, $\dim_H(A)$ is not affected by translation. This is slightly less elementary than (i)-(vii), but is easily verified on noting that if $2^n \gg |x|$, then $x + S_n \subseteq S_{n-1} \cup S_{n+1}$.

One consequence of (viii) is worth noting. In the definition of m_h, the 'base point' was chosen to be the origin. Let $x \in \mathbb{Z}^d$, and write $m_h(x, A) = \sum_n \nu_h(x, A, 2^n)$, where $\nu_h(x, A, 2^n)$ is defined by (4), but with $A \cap S_n$ replaced by $A \cap (x + S_n)$. Then (viii) shows that $m_h(0, A) < \infty$ if and only if $m_h(x, A) < \infty$, and in particular that $\dim_H(A)$ is not affected by the choice of base point.

3. A lower bound for ν_h

It is usually easy to obtain good upper bounds on ν_h (and so m_h) by inspection, and in many cases the 'obvious' covering of $A \cap S_n$ is essentially optimal. However, it is generally quite tricky to prove this optimality directly, since this means considering all coverings of $A \cap S_n$.

We now give a result which gives a lower bound on ν_h. This is a discrete analogue of the density lemma of [5].

Theorem 1. Let $A \subseteq S_N$ and μ be a measure on A. If for some $K > 0$ and all $x \in \mathbb{Z}^d, 0 \leq n \leq N$,

$$\mu(A \cap C(x, 2^n)) \leq K h(2^{n-N}) \tag{7}$$

then

$$\nu_h(A, 2^N) \geq 2^{-d} K^{-1} \mu(A). \tag{8}$$

Proof. Let (Q_i) be an optimal covering of $A \cap S_N$ by dyadic cubes, and write $s(Q_i) = 2^{n_i}$. Then, as the (Q_i) are disjoint,

$$\tilde{\nu}_h(A, 2^N) = \sum_i h(2^{n_i - R}) \geq \sum_i K^{-1} \mu(A \cap Q_i) = K^{-1} \mu(A).$$

Equation (8) now follows on using (6).

4. Examples

We now calculate the discrete Hausdorff dimension of some illustrative specimen sets in $\mathbb{Z}^d$.

4.1. A k-dimensional hyperplane

Let $k \leqslant d$ and set

$$H_k = \{x = (x_1, \ldots, x_d) \in \mathbb{Z}^d : x_{k+1} = \cdots = x_d = 0\}.$$

By property (vi), $\dim_H(H_k) \leqslant \dim_{UM}(H_k) = k$. Let $\alpha \leqslant k$, let $N \geqslant 1$ and let μ be the measure which assigns mass 1 to each point in $H_k \cap S_N$. Then for $0 \leqslant n \leqslant N$

$$\mu(H_k \cap C(x, 2^n)) \leqslant 2^{nk} \leqslant 2^{Nk}(2^{n-N})^\alpha.$$

So (7) holds with $K = 2^{Nk}$, $h(x) = x^\alpha$. Hence, by (8), and as $\mu(H_k \cap S_N) = 2^{Nk}(1 - 2^{-k})$,

$$\nu_K(H_K, 2^N) \geqslant 2^{-d}(1 - 2^{-k}).$$

Thus $m_\alpha(H_K) = \infty$ for $\alpha \leqslant k$, which implies that $\dim_H(H_k) = k$.

4.2. Thinly spaced sets

Let $0 < \alpha < d$, and define T_α by taking $T_\alpha \cap S_n$ to be $2^{\alpha n}$ points uniformly spaced over the set S_n. Since

$$|T_\alpha \cap V(0, 2^n)| = \sum_{i=1}^{n} 2^{\alpha i} \leqslant c_1 2^{\alpha n}$$

property (vi) implies that $\dim_H(T_\alpha) \leqslant \alpha$.

Let μ be the measure which assigns mass 1 to each point in $T_\alpha \cap S_N$, so that $\mu(T_\alpha \cap S_N) = 2^{\alpha N}$, and let $\beta \leqslant \alpha$. Then, as T_α has density $2^{N(\alpha - d)}$ in S_N,

$$\mu(T_\alpha \cap C(x, 2^n)) \leqslant c_2 \max(1, 2^{nd + N\alpha + Nd}).$$

It is easily verified from this that

$$\mu(T_\alpha \cap C(x, 2^n)) \leqslant c_2 2^{\alpha N}(2^{N-n})^\beta$$

and so, by (8),

$$\nu_\beta(T_\alpha, 2^N) \geqslant c_2.$$

Hence $m_\beta(T_\alpha) = \infty$ for $0 \leqslant \beta \leqslant \alpha$, and $\dim(T_\alpha) = \alpha$.

For both these examples the discrete Hausdorff and mass dimensions are the same. Our third example (based on a well known example in potential theory) shows that this is not always the case.

4.3. A sequence of fat cubes

Let $z = (1, 0, \ldots, 0) \in \mathbb{Z}^d$, $\alpha > 0$, $a_n = (2 + n)^{-\alpha} 2^n$ and set

$$F_\alpha = \bigcup_{n=1}^{\infty} C(2^n z, a_n). \tag{9}$$

Clearly $\dim_M(D_\alpha) = d$. However, covering $F_\alpha \cap S_N$ by the cube $C(2^{N-2}z, a_{N-2})$ gives, for $0 < \beta \leqslant d$,

$$\nu_\beta(F_\alpha, 2^N) \leqslant 2^{-2\beta} N^{-\alpha\beta}.$$

Hence (using (vi)) we have $\dim_H(F_\alpha) \leqslant \min(\alpha^{-1}, d)$. A calculation similar to that in the previous two examples shows that, in fact, $\dim_H(F_\alpha) = \min(\alpha^{-1}, d)$.

If instead we take $a_n = 2^{n\gamma}$, with $0 \leq \gamma < 1$, we obtain, writing G_γ for the set defined by (9),

$$\dim_M(G_\gamma) = \gamma d \qquad \dim_H(G_\gamma) = 0.$$

The reason that the discrete Hausdorff dimension of the sets F_α, G_γ is smaller than the mass dimension is that these sets are highly concentrated. Most of the points in these sets are 'wasted'.

In all of these examples the lower bound on ν_α was obtained by using theorem 1 with μ as counting measure on $A \cap S_N$. For more complicated sets, another choice of μ may be necessary.

4.4. Remarks on Naudts's definition

An alternative definition of dimension in $\mathbb{Z}^d$ is given in [4]. We will refer to this as $\dim_N$. This definition seems to us to be seriously flawed: it is not monotone.

An example is as follows. Let $0 < \alpha < d$, $\gamma = \alpha/d$, and consider the sets T_α, G_γ defined above. Then $\dim_N(T_\alpha) = \alpha$ (all the definitions agree here), while $\dim_N(G_\gamma) = d$. (Here $\dim_N$ seeks out the thickest part of the set.) Now

$$|T_\alpha \cap V(0, 2^n)| \simeq |G_\gamma \cap V(0, 2^n)| \simeq 2^{n\alpha}$$

and so $m_\beta(T_\alpha \cup G_\gamma, s) \geq c m_\beta(T_\alpha, s)$ for some $c > 0$. (Here m_β refers to the quantity defined by equation (2) in [4].) From this it follows that

$$\dim_N(T_\alpha \cup G_\gamma) \leq \alpha < \dim_N(G_\gamma) = d.$$

5. Connections with random walks

There is a well known link between the (usual) Hausdorff dimension of a set A in $\mathbb{R}^d (d \geq 3)$ and Brownian motion: if $\dim(A) < d - 2$ then, with probability 1, A is not hit by Brownian paths, while if $\dim(A) > d - 2$ then A is hit with positive probability.

We now announce an analogous result in $\mathbb{Z}^d$. The analogue of Brownian motion is the nearest-neighbour random walk $X = (X_n, n \geq 0)$. Plainly, any non-empty set is hit by X with positive probability: the correct analogue is whether a set is hit by X infinitely often or not. (A zero-one law states that, if $A \subseteq \mathbb{Z}^d$, then $\pi_\infty(A) =$ prob (X hits A infinitely often) is either 0 or 1. If $\pi_\infty(A) = 1$, A is recurrent, otherwise A is transient: see [6, 7].)

Theorem 2. Let $A \subseteq \mathbb{Z}^d$, where $d \geq 3$.

(a) If $m_{d-2}(A) < \infty$ then A is transient.

(b) If $\dim_H(A) > d - 2$ then A is recurrent.

Remarks

(i) Note that $\dim_H(A) < d - 2$ implies that $m_{d-2}(A) < \infty$.

(ii) If $\dim_H(A) = d - 2$ and $m_{d-2}(A) = \infty$ then A may be either recurrent or transient: discrete Hausdorff dimension is not a sensitive enough measure of size to resolve these critical cases.

(iii) This result goes some way to justifying the value for the discrete Hausdorff dimension of the set F_α defined by (10).

The proof of theorem 2 will appear in [8].

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